Solving Nonlinear Partial Differential Transport-Density Equation with Diffusion Traffic Flow Model using the Generalized Hyperbolic Functions Method (GHFM) and the First Integral Method (FIM)

Rfaat Soliby, Fazlina Aman

Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia, Pagoh Campus, Malaysia

fazlina@uthm.edu.my

ABSTRACT

This research is about solving nonlinear partial differential transport-density equation with diffusion traffic flow model describing the evolution of the macroscopic quantities traffic density ρ, traffic speed V and traffic flow rate Q, by obtaining exact travelling wave solutions using first integral method (FIM) and generalized hyperbolic functions methods (GHFM). A family of new exact solutions are formally presented. Unlike the Lighthill, Whitham and Richards known as the LWR model, velocity was considered as a dependent variable of density $v(\rho)$ and by adding the viscosity term so as to smooth the resulting density field; however viscosity term could be interpreted in traffic flow model as the resistance of vehicles' drivers to change their velocity instantaneously.

Keywords: *Exact solution, First Integral Method, Generalized Hyperbolic Functions Method, Traffic flow model.*

I. INTRODUCTION

Recently, traffic congestion has been increasing all over the world, consequently, more fuel wastage, monetary losses and environmental pollution. When traffic jam exists queue of vehicles appears, thus our burning need is to know the estimated time to reduce the queue, and how they will spread in location and time. To tackle this problem, individuals must know more about the factors those play an immense rule in traffic jam mainly the traffic flow and density on the roads. Up to this time, a lot of attempts have been done to figure out the traffic flow issue, for instance Hartono, Binatari and Saptaningtyas [1] built their model based on Greenshield's model to solve the problem of perturbation in the velocity value by using multiple scale method. Meanwhile, Liu et al. [2] constructed their statistical entropy model based on the proportion of the distance between the cars, and considered it as the data for entropy calculation, nevertheless, by taking vehicle's speed into consideration the more change in the car's velocity the more entropy value calculated by the traffic flow. Burger, Göttlich and Jung [3] on the other side, they adjusted the LWR model by adding an explicit delay time to the flow function finally the result was a first order delayed macroscopic traffic flow model.

Assuming that the numbers of cars are conserved and by using the hydrodynamic flow relation it is evident that the first kinematic macroscopic model introduced by Lighthill and Whitham [4] plays a role in controlling the traffic. For a long time LWR model was considered one of the best description for the flux, by the time passing it appeared that it is not sufficient enough to analysis the data collected from the highways due to too many unrealistic assumptions, for example, the car velocity adapts immediately to the desired velocity, the shock wave is not realistic in real situation, discontinuity for some functions, limitless acceleration, dispersion and so on. The fact that shocks still exist in the inviscid models has led some scientists to create an even more sophisticated description of traffic model and this has been studied by Zhang [5]. So that, a small viscosity term was added to the Payne-Whitham model to obtain the viscous model prepared by Kuhne [6]. Therefore, a lot of publications have been produced about the viscous model like the researches done by Kerner and Konhäuser [7]. Remarkably, adding viscosity term has been the most important concept in all researches. Li et al. [8], solved most of these problems by using artificial viscosity because in order to reflect the fact that drivers will decrease their speed when increasing density ahead assuming that the flow Q is a function of the density gradient. Meanwhile, Cathleen Perlman [9] used the Fick's law of diffusion to adjust the so-called continuity equation and as a consequence lots of shortcomings had been solved with the new diffusion form.

II. PROBLEM FORMULATION

The transport-density equation was derived by using the Greenshield [10] constitutive law of velocity $v = v_m - \frac{v_m}{\rho_m}\rho$, and the hydrodynamic flow relation $Q = \rho \times v$ where the velocity is a function of traffic density $v(\rho)$ and apply that to the new form after adding viscosity term, to the continuity equation,

$$\rho_t + \left(Q(\rho) - d.\rho_x\right)_x = 0 \cdot \qquad (1)$$

Given that, $\frac{dQ}{d\rho} = Q'(\rho) = v_m - \frac{2v_m}{\rho_m}\cdot\rho$, which known as group velocity of a traffic wave, meanwhile $v(\rho)$ is the vehicle velocity as individual. Consequently, $\rho_t + Q'\rho_x - d\cdot\rho_{xx} = 0$ and finally the nonlinear transport-density equation would take the form,

$$\rho_t + \left[v_m - \frac{2v_m}{\rho_m}\cdot\rho\right]\cdot\rho_x - d\cdot\rho_{xx} = 0 \cdot \qquad (2)$$

Feng [11] was the first one who formulated and introduced the first integral method which mostly built on the theories from commutative algebra.

First, let's introduce the wave transformation $\rho(x,t) = f(\varepsilon) = f(kx - \omega t)$ where $f(\varepsilon)$ is an arbitrary function of the characteristic variable ε, k and ω are constants, by applying this transformation on Eqn. (2) the result is

$$dk^2 f'' = \left[-\omega + v_m \cdot k\right]\cdot f' - \left[\frac{2k\cdot v_m}{\rho_m}\right] f\cdot f' \cdot$$

As long as $d\cdot k^2 \neq 0$;

$$f'' = \left[\frac{-\omega + v_m k}{d\cdot k^2}\right] f' - \left[\frac{2v_m}{d\cdot k\rho_m}\right] f\cdot f' \cdot$$

For simplicity, rename c_1 and c_2 as $c_1 = \left[\frac{-\omega + v_m k}{d\cdot k^2}\right]$ and $c_2 = \left[\frac{2v_m}{d\cdot k\cdot\rho_m}\right]$ respectively, where $c_1, c_2 \in \mathbb{R}$.

$$f'' = c_1 f' - c_2 f\cdot f' \cdot \qquad (3)$$

Subsequently, partial differential equation (PDE) Eqn. (2) is then transformed to ordinary differential equation (ODE) Eqn. (3).

Now, presenting some new independent variables,

$$\begin{cases} X(\varepsilon) = f(\varepsilon), \\ Y(\varepsilon) = \frac{\partial f(\varepsilon)}{\partial\varepsilon}. \end{cases} \qquad (4)$$

This results a nonlinear system of ODE as follow,

$$\begin{cases} \frac{\partial X}{\partial\varepsilon} = Y, \\ \frac{\partial Y}{\partial\varepsilon} = c_1 Y - c_2 X\cdot Y. \end{cases} \qquad (5)$$

It is an ongoing quest to seek a solution for the system (5). Therefore, an assumption that the functions $X(\varepsilon)$ and $Y(\varepsilon)$ are nontrivial solutions of system (5) is required.

Hosseini, Ansari and Gholamin [12], took advantage from the qualitative theory of ordinary differential equations and they referred if individual can find the integrals to the system (5) under the same conditions, then the general solutions can be obtained directly. With the help from Hilbert–Nullstellensatz Theorem and the (1) Division Theorem in order to find one first integral to the system (5) which will reduce the system to a first order integrable ordinary differential equation one can obtain a new exact solution Feng [13].

Now, let's assume that, $R(X,Y)$ is an irreducible polynomial in the complex domain *C [X, Y]* as follow,

$$R(X,Y) = \sum_{i=0}^{N} a_i(X)\cdot Y^i = 0. \qquad (6)$$

Eqn. (6) is considered as the first integral to the system (5), where $a_i(X)$ are polynomials of X such that $a_N(X) \neq 0$. Therefore, according to the Division theorem, a polynomial $g(X) + h(X)\cdot Y$ can be found in the complex domain *C[X, Y]*, in the form,

$$\frac{dR}{d\varepsilon} = \frac{\partial R}{\partial X}\cdot\frac{\partial X}{\partial\varepsilon} + \frac{\partial R}{\partial Y}\cdot\frac{\partial Y}{\partial\varepsilon} = \left[g(X) + h(X)Y\right]\cdot\sum_{i=0}^{N} a_i(X)\cdot Y^i. \qquad (7)$$

Suppose $N = 1$:

Eqn. (6) becomes,

$$R(X,Y) = a_0(X) + a_1(X)\cdot Y = 0, \qquad (8)$$

and the left-hand side of Eqn. (7) becomes,

$$\frac{dR}{d\text{E}} = \left[a_0'(X) + a_1'(X)Y\right]\cdot Y + \left[a_1(X)\right]\cdot Y',$$

or,

$$\frac{dR}{d\text{E}} = \left[a_0'(X) + c_1 a_1(X) - c_2 a_1(X)X\right]Y + a_1'(X)Y^2, \qquad (9)$$

and the right-hand side of Eqn. (7) becomes,

$$\frac{dR}{d\text{E}} = \left[g(X) + h(X)Y\right]\cdot\left[a_0(X) + a_1(X)Y\right],$$

or,

$$\frac{dR}{d\text{E}} = g(X)a_0(X) + \left[g(X)a_1(X) + h(X)a_0(X)\right]Y + h(X)a_1(X)Y^2. \qquad (10)$$

Next, Comparing Eqn. (9) and Eqn. (10) with the coefficients of Y^i for $i = 0, 1, 2$ leads to,

Coefficients of Y^2 : $a_1'(X) = h(X) \cdot a_1(X)$. (11)

Coefficients of

Y : $a_0'(X) + c_1 a_1(X) - c_2 a_1(X) \cdot X = g(X) a_1(X) + h(X) a_0(X)$.

Coefficients of Y^0 : $g(X) \cdot a_0(X) = 0$.

(12)

Hence, $a_1(X)$ and $a_0(X)$ are polynomials of X , it can be concluded from Eqn. (11) that $a_1(X)$ is a constant and $h(X) = 0$. For easiness, let's take $a_1(X) = 1$. Now, by balancing the degrees for $g(X)$ and $a_0(X)$ it can be said that deg $[g(X)] = 0$ or 1.

Assume that deg $[g(X)]$ = 0 which means $g(X) = B$ where B is a constant. So,

$a_0'(X) + c_1 - c_2 X = B$ or $a_0'(X) = c_2 X - c_1 + B$.

Integrating once leads to,

$$a_0(X) = \frac{c_2}{2} X^2 + (B - c_1) X + A. \quad (13)$$

Replacing $g(X) = B$ and $a_0(X)$ in Eqn. (12) and setting all the coefficients of powers X to be equal zero, will result

$B = 0$ and $A \neq 0$ so Eqn. (13) becomes,

$$a_0(X) = \frac{c_2}{2} X^2 - c_1 X + A. \quad (14)$$

Substituting Eqn. (14) and $a_1(X) = 1$ in Eqn. (8) to get,

$$Y = \frac{dX}{d\varepsilon} = \frac{-c_2}{2} X^2 + c_1 X - A,$$

$$\frac{dX}{d\varepsilon} = \frac{-c_2}{2}\left(X^2 - \frac{2c_1}{c_2} X + \frac{2A}{c_2} \right),$$

$$\frac{1}{\left(X - \frac{c_1 + \sqrt{c_1^2 - 2Ac_2}}{c_2} \right)\left(X - \frac{c_1 - \sqrt{c_1^2 - 2Ac_2}}{c_2} \right)} dX = \frac{-c_2}{2} d\varepsilon,$$

$$\frac{1}{(X - X_r)(X - X_l)} dX = \frac{-c_2}{2} d\varepsilon.$$

So,

$\varepsilon = \dfrac{2}{-c_2 (X_r - X_l)} \ln \dfrac{X_r - X}{X - X_l}$; Given that $X_r > X > X_l$ and

$$X_r = \frac{c_1 + \sqrt{c_1^2 - 2Ac_2}}{c_2}, \quad X_l = \frac{c_1 - \sqrt{c_1^2 - 2Ac_2}}{c_2}.$$

Using Maple,

$$X = \frac{X_r + X_l e^{-\sqrt{c_1^2 - 2Ac_2}.\varepsilon}}{e^{-\sqrt{c_1^2 - 2Ac_2}.\varepsilon} + 1},$$

$$X = \frac{\frac{c_1 - \sqrt{c_1^2 - 2Ac_2}}{c_2} + \frac{c_1 + \sqrt{c_1^2 - 2Ac_2}}{c_2} e^{\sqrt{c_1^2 - 2Ac_2}.(kx - \omega t)}}{e^{\sqrt{c_1^2 - 2Ac_2}.(kx - \omega t)} + 1}.$$

Using Maple and the system (4), to obtain a family of new exact solutions for Eqn. (2).

Or hyperbolically,

$$\rho(x,t) = \frac{c_1}{c_2} + \frac{\sqrt{c_1^2 - 2Ac_2}}{c_2} tanh\left(0.5\sqrt{c_1^2 - 2Ac_2}.(kx - \omega t)\right),$$

(15)

where, $2Ac_2 < c_1^2$.

III. THE GENERALIZED HYPERBOLIC FUNCTION METHOD (GHFM)

The analogy of traffic flow to density has been studied too much before. But some imperative differences occur, e.g. relaxation time, viscosity, and till what degree conservation law stands. Lots of researchers have attempted hard to solve the nonlinear PDEs using the generalized hyperbolic function method (GHFM) by taking different formulas to construct a new exact solution. Extraordinarily, it was first proposed by Gao [14] then by many others like Al-Muhiameed & Abdel-Salam [15], and Wazzan [16].

Devotedly, new solutions were built depending on $tanh_r, sech_r$ which satisfy many relations quite similar to those for the $tanh, sech$ functions.

The generalized hyperbolic cosecant function which defined as:

$$sech_r(\lambda) = \frac{2}{p \cdot r^{m\lambda} + q \cdot r^{-m\lambda}},$$

and the generalized hyperbolic cotangent function which defined as:

$$tanh_r(\lambda) = \frac{p \cdot r^{m\lambda} - q \cdot r^{-m\lambda}}{p \cdot r^{m\lambda} + q \cdot r^{-m\lambda}}.$$

Accordingly, the derivatives formulas would take the form,

$$\frac{d(tanh_r(\lambda))}{d\lambda} = m\, pq \ln r sech_r^2(\lambda),$$

and obviously $sech_r$ and $tanh_r$ satisfy:

$$tanh_r^2(\lambda) = 1 - pqsech_r^2(\lambda).$$

All the above symmetrical hyperbolic Fibonacci and Lucas functions were built and proved well by Pandir and Ulusoy [17].

Notice that when, $\begin{cases} p = 1 \\ q = 1 \\ m = 1 \\ r = e \end{cases}$

that will lead us to normal hyperbolic functions $tanh, sech$.

So, let's start solving transport-density equation,

$$\rho_t + v_m \rho_x - \frac{2v_m}{\rho_m} \cdot \rho \cdot \rho_x - d \cdot \rho_{xx} = 0.$$

By taking the same wave transformation: $\rho(x,t)=\rho(\varepsilon)\;:\varepsilon=kx-\omega t$,

$$-\omega\rho'+v_m k\rho'-\frac{2v_m k}{\rho_m}\rho\rho'-k^2 d\cdot\rho''=0.$$

Integrating once in term of ε :

$$-\omega\rho+v_m k\rho-\frac{v_m k}{\rho_m}\rho^2-k^2 d\cdot\rho'=0,$$

(16)

(integrating constant = 0).

The solution was built by using the generalized hyperbolic tangent function in the form,

$Z=tanh_r(\mu\varepsilon)$, where μ, r are free parameters.

The hypothesis is that

$$\rho(\varepsilon)=S(Z)=\sum_{i=0}^{N}a_i Z^i, \tag{17}$$

is a solution for transport density equation. So that, the objective is to determine the constants a_i for $i=0,1,2,\ldots,N<+\infty$, with $a_N\neq 0$ those who satisfy the Eqn. (16).

Balancing the linear term of highest order of equation Eq. (16) with the highest order nonlinear term to determine the value of N, so by balancing ρ' with ρ^2 leads to the equation $N+1=2N$ so, $N=1$, by replacing in Eqn. (17), to finally have the form,

$$\rho(\varepsilon)=a_0+a_1 Z\;;\;Z=tanh_r(\mu\varepsilon).$$

Now by replacing in Eqn. (16), one can get

$$C_0+C_1 Z+C_2 Z^2=0, \tag{18}$$

where,

$$C_0=-\omega a_0+v_m k a_0-\frac{v_m k}{\rho_m}a_0^2-k^2 d a_1\mu\ln r,$$

$$C_1=-\omega a_1+v_m k a_1-\frac{2v_m k}{\rho_m}a_0 a_1,$$

$$C_2=-\frac{v_m k}{\rho_m}a_1^2+k^2 d a_1\mu\ln r.$$

By substituting the above results in Eqn. (18) collecting the coefficients of each power of $\tanh_a^i(\mu\varepsilon)$ for $i=0,1,2$ and setting each coefficient to be zero and by using Maple two cases were produced,

Case 1

$$a_0=\frac{kd\mu\rho_m\ln r}{v_m},\quad a_1=\frac{kd\mu\rho_m\ln r}{v_m},\omega=v_m k-2k^2 d\mu\ln r.$$

Substituting these coefficients in Eqn. (17),

$$\rho(x,t)=\frac{kd\mu\rho_m\ln r}{v_m}+\frac{kd\mu\rho_m\ln r}{v_m}tanh_r\left(k\mu x-\mu\left(v_m k-2k^2 d\mu\ln r\right)t\right).$$

Case 2

$$a_0=-\frac{kd\mu\rho_m\ln r}{v_m},\quad a_1=\frac{kd\mu\rho_m\ln r}{v_m},\quad \omega=v_m k+2k^2 d\mu\ln r.$$

Substituting these coefficients in Eqn. (17),

$$\rho(x,t)=-\frac{kd\mu\rho_m\ln r}{v_m}+\frac{kd\mu\rho_m\ln r}{v_m}tanh_r\left(k\mu x-\mu\left(v_m k+2k^2 d\mu\ln r\right)t\right).$$

IV. RESULTS AND DISCUSSIONS

Two particular cases are shown here. These approaches regarding the FIM and GHFM were achieved for a given values for the parameters as follow, $d=0.02$, $k=1$, A = 1/20, $\mu=1$ and $r=e$, $\rho_m=200$ cars / km, $v_m=60$ km/h. These parameters must be controlled by the collected data from roads through sufficient number of experiments. Thus, different values of these parameters will unquestionably lead to different density and flow values.

Regarding the FIM and in order to avoid sudden changes in traffic density, drivers try to stay spread out by decreasing the cars' velocity first when they see that the traffic density ahead is increasing, this result due to the diffusion term, so that the transport-density equation has smooth shock profiles modeling transitions from low density ρ_r to high density ρ_l. Thus ρ is a smooth function that increases monotonically from ρ_l to ρ_r.

So, considering the given particular parameters with $\omega=59.96$ and by using the traveling wave solution obtained by FIM, we get,

$$\rho(x,t)=\frac{2}{30}+\frac{1}{30}\tanh(0.5x-29.98t).$$

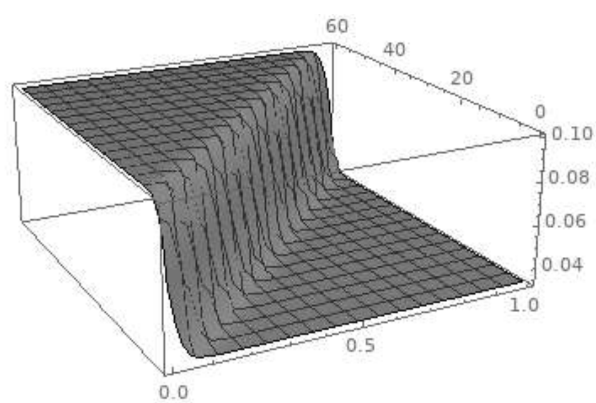

Fig. 1. Illustration of travelling wave solution obtained by the FIM for the particular parameters for the time interval $t\in[0,1]$,road distance $x\in[0,60]$.

Now, for the different times the effects of propagating traffic density is shows in Figure 2, where density considered as a function of distance for three different times equal to 0.1, 0.2, 0.3.

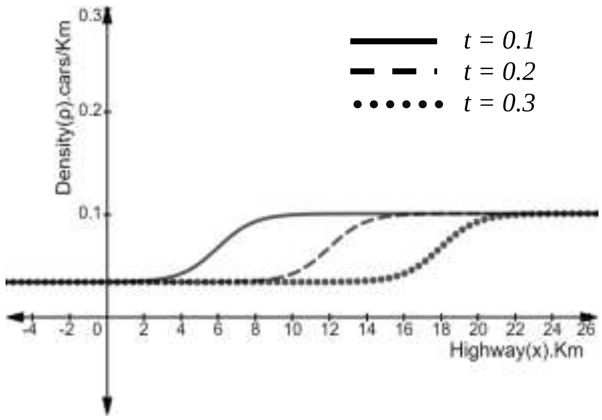

Fig. 2. Profiles of the travelling wave solution of the various times for *t* equal to 0.1, 0.2, and 0.3.

To observe the asymptotic behavior for this particular solution, limits have been taken, so

$$\lim_{x\to+\infty}\rho(x,t)=\frac{c_1+\sqrt{c_1^{\,2}-2Ac_2}}{c_2}\cong 0.1=X_r,$$

$$\lim_{x\to-\infty}\rho(x,t)=\frac{c_1-\sqrt{c_1^{\,2}-2Ac_2}}{c_2}\cong 0.0333=X_l,$$

It is apparent that the wave profile travels from left to right with speed *S* equal to the average of its asymptotic values, $S=\frac{1}{2}(X_r+X_l)=\frac{c_1}{c_2}=\frac{2}{30}$.

The diffusion term *d* play a huge role to prevent the gradual distortion of the wave by countering the nonlinearity. Now, at a fixed time $t=0.1$, consequently, one conclude that the smaller viscosity, the sharper the transition between the two asymptotic values as shown in Figure 3.

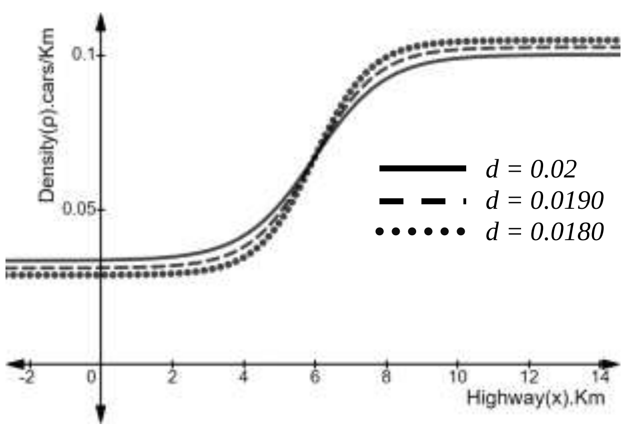

Fig.3. The profiles of the travelling wave solution of the various viscosities for *d* equal to 0.02, 0.0190, and 0.0180.

On the other hand, regarding the GHFM, and by utilizing the given parameters and the traveling wave solution of Case 1 obtained by GHFM, we have,

$$\rho(x,t)=\frac{1}{15}+\frac{1}{15}tanh(x-59.96t).$$

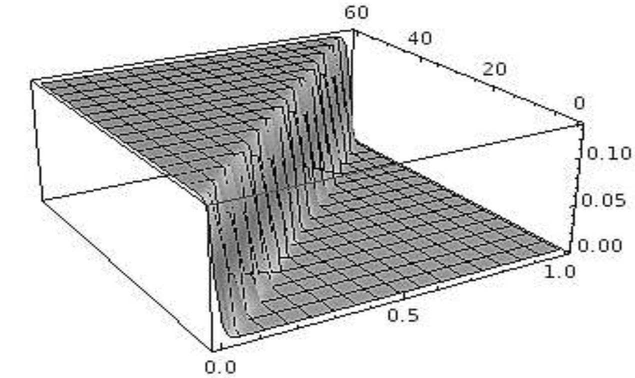

Fig. 4. Illustration of travelling wave solution obtained by the GHFM for the particular parameters and the time interval $t\in[0,1]$, road distance $x\in[0,60]$ and viscosity $d=0.02$.

Now, the effects of propagating traffic density on the traveling wave solutions for the different times is shows in Figure 5, where density is a function of distance.

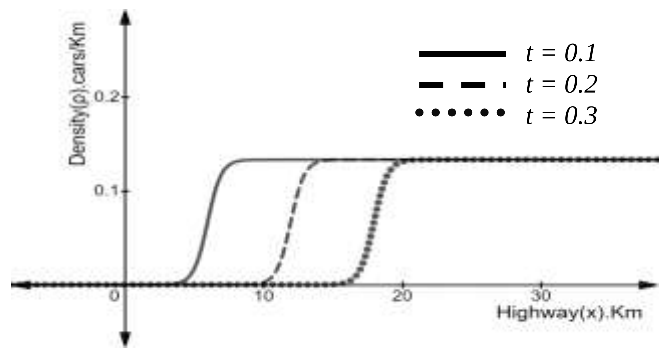

Fig. 5. Profiles of the travelling wave solution of the various times for *t* equal to 0.1, 0.2, and 0.3.

Now let’s vary the viscosity using GHFM at a fixed time $t=0.1$ as shown in Figure 6.

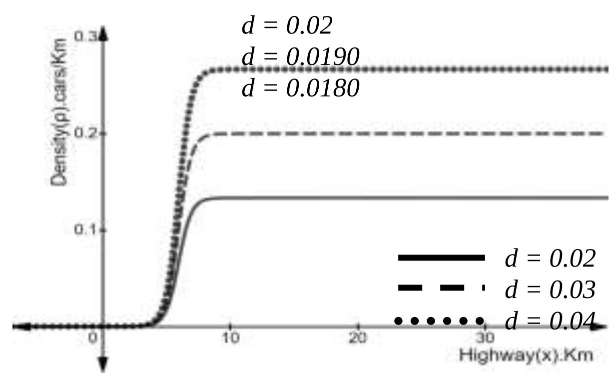

Fig. 6. The profiles of the travelling wave solution using GHFM for various viscosities parameter of *d* equal to 0.02, 0.03, and 0.04.

Obviously, from Figure 6 the larger viscosity, the sharper the transition between the two asymptotic values. To put it briefly, an excellent decision regarding the traffic flow model entirely depends on the parameters which controlled by the information, experiments and observations done on roads. Defining the constants perfectly will undoubtedly lead to outstanding decision making about optimum benefits of using FIM or GHFM

V. CONCLUSION

Through the proposed model, two purely different methods have been done in order to find new exact solutions for the nonlinear partial differential transport-density equation. However, physically the problem of abrupt change of car's velocity on the roads was solved by adding diffusion term which plays a significant role to encounter both discontinuity and shock formation. Accordingly, smoothing the density profile has been fulfilled successfully. The first technique was the first integral method (FIM), which depends on the division theorem and Hilbert-Nullstellensatz theorem and it was applied successfully to solve the nonlinear partial differential transport-density equation. Meanwhile, the Generalized Hyperbolic Function Method (GHFM) was also applied productively using symmetrical hyperbolic Fibonacci and Lucas functions so that new many families of exact travelling wave solutions are found.

Under the terms of this information, we believe that (FIM) and (GHFM) are powerful mathematical tools to obtain new exact reliable solutions for a wide variety of nonlinear partial differential equations and have several applicability in mathematical physics.

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